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NOTE

A New Proof of the Arithmetic Mean—The Geometric Mean Inequality

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Recently J. Sándor and V. E. S. Szabó [1] have used the elementary inequality

$$\sum_{i=1}^{n} \inf_{x\in E} f_i(x) \le \inf_{x\in E} \sum_{i=1}^{n} f_i(x), \tag{1}$$

where $f_i : E \subset \mathbf{R} \to \mathbf{R}$ $(i = 1, 2, \ldots, n)$ in the proof of some known inequalities. In fact, they said that their inequalities

$$\prod_{i=1}^{n} \left(\frac{b_i}{a_i}\right)^{b_i} \ge \left(\frac{\sum_{i=1}^{n} b_i}{\sum_{i=1}^{n} a_i}\right)^{\sum_{i=1}^{n} b_i} \tag{2}$$

and

$$\frac{\sum_{i=1}^{n} b_i}{n} \le \prod_{i=1}^{n} b_i^{b_i / \sum_{i=1}^{n} b_i} \tag{3}$$

where a_i, $b_i \ge 0$ $(i = 1, 2, \ldots, n)$ are new.

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Moreover, it is easy to show that (2) is equivalent to the arithmetic mean—the geometric mean inequality

$$G_n = \prod_{i=1}^{n} a_i^{p_i / \sum_{i=1}^n p_i} \leq \frac{\sum_{i=1}^n p_i a_i}{\sum_{i=1}^n p_i} = A_n. \tag{4}$$

Namely, setting in (2), $b_i := p_i$, $a_i := p_i a_i$ we shall get (4); and the reverse, setting in (4), $a_i := a_i / b_i$, $p_i := b_i$ we shall get (2).

So their proof of (2) can be regarded as a new proof of the AG inequality but we note that we can use a simpler function for that.

Namely, choose $f_i(x) = p_i(a_i x - \log x)$, p_i, $a_i > 0$. We have $f'(x) = p_i(a_i - 1/x)$ and therefore f_i has minimum value at $x_{i,0} = 1/a_i$ and its value is $f_i(x_{i,0}) = p_i(1 + \log a_i)$. Similarly, we obtain that the function $f(x) = \sum_{i=1}^n f_i(x)$ has minimum at $x_0 = A_n^{-1}$ and its value is $f(x_0) = \sum_{i=1}^n p_i(1 + \log A_n)$. Using (1) we obtain

$$\sum_{i=1}^{n} p_i + \sum_{i=1}^{n} p_i \log a_i \leq \sum_{i=1}^{n} p_i + \sum_{i=1}^{n} p_i \log A_n,$$

i.e.,

$$\sum_{i=1}^{n} p_i \log a_i \leq \sum_{i=1}^{n} p_i \log A_n$$

which is equivalent to the AG inequality.

REFERENCES

1. J. Sándor and V. E. S. Szabó, On an inequality for the sum of infimums of functions, *J. Math. Anal. Appl.* **204** (1996), 646–654.